# ON THE RADICALS OF A GROUP THAT DOES NOT HAVE THE INDEPENDENCE PROPERTY

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Abstract. We give an example of a pure group that does not have the independence property, whose Fitting subgroup is neither nilpotent nor definable and whose soluble radical is neither soluble nor definable. This answers a question asked by E. Jaligot in May 2013.

The Fitting subgroup of a stable group is nilpotent and definable (F. Wagner [Wag95]). More generally, the Fitting subgroup of a group that satisfies the descending chain condition on centralisers is nilpotent (J. Derakhshan, F. Wagner [DW97]) and definable (F. Wagner [Wag99, Corollary 2.5], see also [OH13] and [AB14]). The soluble radical of a superstable group is soluble and definable (A. Baudish [Bau90]). Whether this also holds for a stable group is still an open question.

Inspired by [MT12], we provide an example of a pure group that does not have the independence property, whose Fitting subgroup is neither nilpotent nor definable and whose soluble radical is neither soluble nor definable. The proofs require some algebra because we have decided to provide a precise computation of the Fitting subgroup and soluble radical of the group considered.

**Definition 1** (independence property)**.** Let $M$ be a structure. A formula $\varphi(x, y)$ *has the independence property in* $M$ if for all $n \in \omega$, there are tuples $a_1, \ldots, a_n$ and $(b_J)_{J\subset\{1,\ldots,n\}}$ of $M$ such that $\Big(M \models \varphi(a_i, b_J)\Big) \iff i \in J$. $M$ *does not have the independence property* (or is NIP for short) if no formula has the independence property in $M$.

Let $\mathbf{L}$ be a first order language, $M$ an $\mathbf{L}$-structure. A set $X$ is *interpretable* in $M$ if there is a definable subset $Y \subset M^n$ in $M$ and a definable equivalence relation $E$ on $X$ such that $X = Y/E$. A family $\{Y_i/E_i : i \in I\}$ of interpretable sets in $M$ is *uniformly interpretable in* $M$ if the corresponding families $\{Y_i : i \in I\}$ and $\{E_i : i \in I\}$ are uniformly definable in $M$.

Let $L$ be yet another first order language. An $L$-structure $N$ is *interpretable* in $M$ if its domain, functions, relations and constants are interpretable sets in $M$. A family of $L$-structures $\{N_i : i \in I\}$ is *uniformly interpretable* in $M$ if the family of domains is uniformly interpretable in $M$, as well as, for each symbol $s$ of the language $L$, the family $\{s_i : i \in I\}$ of interpretations of $s$ in $N_i$.

**Lemma 2** (D. Macpherson, K. Tent [MT12])**.** *Let $M$ be an $\mathbf{L}$-structure that does not have the independence property and let $\{N_i : i \in I\}$ be a family of $L$-structures that is uniformly interpretable in $M$. For every ultrafilter $\mathcal{U}$ on $I$, the $L$-structure $\prod_{i\in I} N_i/\mathcal{U}$ does not have the independence property.*

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**Corollary 3.** *Let $m$ and $n$ be natural numbers and $p$ a prime number. Let us consider the general linear group $\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})$ over the finite ring $\mathbf{Z}/p^n\mathbf{Z}$. Let $\mathcal{U}$ be an ultrafilter on $\mathbf{N}$ and let $G$ be the ultraproduct*

$$G = \prod_{n\in\mathbf{N}} \mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z}) \Big/ \mathcal{U}.$$

*The pure group $G$ does not have the independence property.*

*Proof.* Consider the group $\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}_p)$ over the ring $\mathbf{Z}_p$ of $p$-adic integers, and the normal subgroups $\mathbf{1} + p^n\mathrm{M}_{\mathrm{m}}(\mathbf{Z}_p)$ for every $n \geqslant 1$. One has the group isomorphism

$$\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z}) \simeq \mathrm{GL}_{\mathrm{m}}(\mathbf{Z}_p) \Big/ \mathbf{1} + p^n\mathrm{M}_{\mathrm{m}}(\mathbf{Z}_p).$$

Therefore, the family of groups $\{\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z}) : n \in \mathbf{N}\}$ is uniformly interpretable in the ring $\mathrm{M}_{\mathrm{m}}(\mathbf{Q}_p)$, which is interpretable in the field $\mathbf{Q}_p$ of $p$-adic numbers, hence NIP by [Mat93]. By Lemma 2, the group $G$ does not have the independence property. □

## 1. Preliminaries on the normal structure of $\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})$

Given a field $k$, the normal subgroups of the general linear group $\mathrm{GL}_{\mathrm{m}}(k)$ are precisely the subgroups of the centre and the subgroups containing the special linear group $\mathrm{SL}_{\mathrm{m}}(k)$ (J. Dieudonné [Die55]). In particular, the maximal normal soluble subgroup of $\mathrm{GL}_{\mathrm{m}}(k)$ is the centre, except for the two soluble groups $\mathrm{GL}_2(\mathbf{F}_2)$ and $\mathrm{GL}_2(\mathbf{F}_3)$. The situtation is different for the general linear group $\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})$ over the ring $\mathbf{Z}/p^n\mathbf{Z}$, whose normal subgroups are classified by J. Brenner [Bre38]. We follow also W. Klingenberg [Kli60] who deals with the normal subgroups of the general linear group over a local ring $R$, which applies in particular to $\mathbf{Z}/p^n\mathbf{Z}$.

The centre of $\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})$ is the subgroup of homotheties $(\mathbf{Z}/p^n\mathbf{Z})^\times \cdot \mathbf{1}$. The *general congruence subgroup of* $\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})$ *of order* $\ell$ is

$$\mathrm{GC}_{\mathrm{m}}(\ell) = (\mathbf{Z}/p^n\mathbf{Z})^\times \cdot \mathbf{1} + p^\ell\mathrm{M}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z}).$$

It is a normal subgroup of $\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})$. For every element $g$ of $\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})$, there is a maximal $\ell \leqslant n$ such that $g$ belongs to $\mathrm{GC}_{\mathrm{m}}(\ell)$. We call $\ell$ the *congruence order* of $g$.

The *special linear subgroup* of $\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})$ of matrices having determinant 1 is written $\mathrm{SL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})$. An *elementary transvection* is an element of $\mathrm{SL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})$ of the form $\mathbf{1} + re_{ij}$ for $r \in \mathbf{Z}/p^n\mathbf{Z}$ and $i \neq j$. A *transvection* is a conjugate of an elementary transvection.

**Proposition 4** (J. Brenner [Bre38, Theorem 1.5])**.** *Let $\tau$ a transvection of congruence order $\ell$. The normal subgroup of* $\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})$ *generated by $\tau$ is*

$$\left\langle \tau^{\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})} \right\rangle = \mathrm{SL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z}) \cap \left(\mathbf{1} + p^\ell\mathrm{M}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})\right).$$

**Theorem 5** (J. Brenner [Bre38])**.** *Let $mp \geqslant 6$ and $g$ an element of* $\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})$ *of congruence order $\ell$. The normal subgroup* $\left\langle g^{\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})} \right\rangle$ *of* $\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})$ *generated by $g$ satisfies*

$$\mathrm{SL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z}) \cap \left(\mathbf{1} + p^\ell\mathrm{M}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})\right) \subset \left\langle g^{\mathrm{GL}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z})} \right\rangle \subset (\mathbf{Z}/p^n\mathbf{Z})^\times \cdot \mathbf{1} + p^\ell\mathrm{M}_{\mathrm{m}}(\mathbf{Z}/p^n\mathbf{Z}).$$

For any real number $x$, we write $\lfloor x \rfloor$ for the *floor of* $x$, that is $\lfloor x \rfloor$ is the greatest integer $k$ such that $k \leqslant x$.

**Lemma 6.** *For any $m \geqslant 2$, $\ell \geqslant 1$ and $n \geqslant 1$, the group $\mathbf{1} + p^\ell \mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})$ is a normal nilpotent subgroup of $\mathrm{GL_m}(\mathbf{Z}/p^n\mathbf{Z})$ of nilpotency class $\left\lfloor \dfrac{n-1}{\ell} \right\rfloor$.*

*Proof.* For every $x$ in $\mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})$, one has

$$(\mathbf{1}+px)^p = \mathbf{1} + \sum_{k=1}^{p} (px)^k C_p^k = \mathbf{1} + p^2 y.$$

It follows that $\mathbf{1} + p\mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})$ is a nilpotent $p$-group. Its iterated centres are

$$\begin{aligned}
Z(H_n) &= (1+p\mathbf{Z}/p^n\mathbf{Z}) \cdot \mathbf{1} + p^{n-1}\mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})\\
Z_2(H_n) &= (1+p\mathbf{Z}/p^n\mathbf{Z}) \cdot \mathbf{1} + p^{n-2}\mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})\\
&\ \vdots\\
Z_{n-2}(H_n) &= (1+p\mathbf{Z}/p^n\mathbf{Z}) \cdot \mathbf{1} + p^{2}\mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})\\
Z_{n-1}(H_n) &= \mathbf{1} + p\mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z}),
\end{aligned}$$

so the nilpotency class of $\mathbf{1} + p\mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})$ is $n-1$ when $n \geqslant 1$. For every natural number $q$ satisfying $n - q\ell \geqslant \ell$, one has

$$Z_q\left(\mathbf{1} + p^\ell\mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})\right) = \left(1 + p^\ell\mathbf{Z}/p^n\mathbf{Z}\right) \cdot \mathbf{1} + p^{n-q\ell}\mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z}),$$

so the greatest $q$ such that the above $q$th centre is a proper subgroup is the greatest $q$ satisfying $n - q\ell > \ell$. As one has

$$n - q\ell > \ell \iff n - 1 - q\ell \geqslant \ell \iff q \leqslant \frac{n-1}{\ell} - 1,$$

this greatest $q$ is precisely $\left\lfloor \dfrac{n-1}{\ell} \right\rfloor - 1$. □

For any real number $x$, we write $\lceil x \rceil$ for the *ceiling of $x$*, that is $\lceil x \rceil$ is the least integer $k$ such that $k \geqslant x$.

**Lemma 7.** *For any $1 \leqslant \ell \leqslant n$ and $m \geqslant 3$, the group $\mathbf{1} + p^\ell\mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})$ is soluble of derived length $\left\lceil \log_2 \dfrac{n}{\ell} \right\rceil$.*

*Proof.* Let us write $\mathrm{PC_m}(\ell) = \mathbf{1} + p^\ell\mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})$ and show that

$$\left(\mathrm{SL_m}(\mathbf{Z}/p^n\mathbf{Z}) \cap \mathrm{PC_m}(\ell)\right)' = \mathrm{PC_m}(\ell)' = \mathrm{SL_m}(\mathbf{Z}/p^n\mathbf{Z}) \cap \mathrm{PC_m}(2\ell).$$

Let $\alpha = \mathbf{1} - p^\ell\gamma$ and $\beta = \mathbf{1} - p^\ell\delta$ be two elements of $\mathbf{1} + p^\ell\mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})$. Then

$$\begin{aligned}
\alpha\beta\alpha^{-1}\beta^{-1} &= \left(\mathbf{1} - p^\ell\gamma\right)\left(\mathbf{1} - p^\ell\delta\right)\left(\mathbf{1} + p^\ell\gamma + \cdots + p^{n\ell}\gamma^n\right)\left(\mathbf{1} + p^\ell\delta + \cdots + p^{n\ell}\delta^n\right)\\
&= \mathbf{1} + p^{2\ell}(\gamma\delta - \delta\gamma) + p^{3\ell}(\cdots) + \cdots,
\end{aligned}$$

so $\mathrm{PC_m}(\ell)'$ is included in $\mathrm{SL_m}(\mathbf{Z}/p^n\mathbf{Z}) \cap \mathrm{PC_m}(2\ell)$. Conversely, consider the two elementary transvections $\sigma = \mathbf{1} + p^\ell e_{12}$ and $\tau = \mathbf{1} + p^\ell e_{21}$. One has

$$\begin{aligned}
\sigma\tau\sigma^{-1}\tau^{-1} &= (\mathbf{1} + p^\ell e_{12})(\mathbf{1} + p^\ell e_{21})(\mathbf{1} - p^\ell e_{12})(\mathbf{1} - p^\ell e_{21})\\
&= \mathbf{1} + p^{2\ell}e_{11} - p^{2\ell}e_{22} - p^{3\ell}e_{12} + p^{3\ell}e_{21}.
\end{aligned}$$

It follows that $\left(\mathrm{SL_m}(\mathbf{Z}/p^n\mathbf{Z})\cap\mathrm{PC_m}(\ell)\right)'$ contains an element that lies in $\mathrm{PC_m}(2\ell)\setminus\mathrm{PC_m}(2\ell+1)$. As $\left(\mathrm{SL_m}(\mathbf{Z}/p^n\mathbf{Z})\cap\mathrm{PC_m}(\ell)\right)'$ is a characteristic subgroup of $\mathrm{SL_m}(\mathbf{Z}/p^n\mathbf{Z})\cap\mathrm{PC_m}(\ell)$, it is normal in $\mathrm{GL_m}(\mathbf{Z}/p^n\mathbf{Z})$. By Theorem 5, $\left(\mathrm{SL_m}(\mathbf{Z}/p^n\mathbf{Z})\cap\mathrm{PC_m}(\ell)\right)'$ contains $\mathrm{SL_m}(\mathbf{Z}/p^n\mathbf{Z})\cap\mathrm{PC_m}(2\ell)$. We have thus shown that for every natural number $k$, the $k$th derived subgroup of $\mathrm{PC_m}(\ell)$ is

$$\mathrm{PC_m}(\ell)^{(k)} = \mathrm{SL_m}(\mathbf{Z}/p^n\mathbf{Z})\cap\mathrm{PC_m}(2^{\mathrm{k}}\ell).$$

The derived length of $\mathrm{PC_m}(\ell)$ is the least $k$ such that $2^k\ell\geqslant n$. □

**Lemma 8.** *For natural numbers $k$ and $n\geqslant k^2+k$, let $\ell_n(k)=1+\left\lfloor\dfrac{n}{k+1}\right\rfloor$. Then $\ell_n(k)$ is the smallest natural number satisfying the equality $\left\lfloor\dfrac{n}{\ell_n(k)}\right\rfloor=k$.*

*Proof.* Let $n=q(k+1)+r$ be the Euclidian division of $n$ by $k+1$, with $q\geqslant k$ and $0\leqslant r<k+1$. Then one has

$$0<\frac{k+1-r}{1+q}\leqslant 1 \quad\text{hence}\quad \left\lfloor\frac{n}{\ell_n(k)}\right\rfloor=\left\lfloor\frac{(k+1)q+r}{1+q}\right\rfloor=\left\lfloor k+1-\frac{k+1-r}{1+q}\right\rfloor=k,$$

so $\ell_n(k)$ satisfies the equality. It is the smallest such, as one has

$$\left\lfloor\frac{n}{\ell_n(k)-1}\right\rfloor=\left\lfloor\frac{n}{\left\lfloor\frac{n}{k+1}\right\rfloor}\right\rfloor=\left\lfloor\frac{n}{q}\right\rfloor=\left\lfloor k+1+\frac{r}{q}\right\rfloor\geqslant k+1. \quad □$$

**Lemma 9.** *For natural numbers $k\geqslant 1$ and $n\geqslant 2^k$, let $d_n(k)=\left\lceil\dfrac{n}{2^k}\right\rceil$. Then $d_n(k)$ is the smallest natural number satisfying the equality $\left\lceil\log_2\dfrac{n}{d_n(k)}\right\rceil=k$.*

*Proof.* One has

$$\frac{n}{2^k}\leqslant\left\lceil\frac{n}{2^k}\right\rceil<\frac{n}{2^k}+1$$

hence

$$k-1\leqslant k-\log_2\left(1+\frac{2^k}{n}\right)<\log_2\left(\frac{n}{\left\lceil\frac{n}{2^k}\right\rceil}\right)\leqslant k,$$

so that $d_n(k)$ satisfies the equality. It is the smallest such, as

$$\left\lceil\log_2\left(\frac{n}{\left\lceil\frac{n}{2^k}\right\rceil-1}\right)\right\rceil=\left\lceil\log_2\left(\frac{2^k}{\frac{2^k}{n}\left\lceil\frac{n}{2^k}\right\rceil-\frac{2^k}{n}}\right)\right\rceil=\left\lceil k-\log_2\left(\frac{2^k}{n}\left\lceil\frac{n}{2^k}\right\rceil-\frac{2^k}{n}\right)\right\rceil\geqslant k+1. \quad □$$

## 2. Radicals of $G$

We now condiser

$$G=\prod_{n\in\mathbf{N}}\mathrm{GL_m}(\mathbf{Z}/p^n\mathbf{Z})\big/\mathcal{U}.$$

We call *Fitting subgroup of $G$* and write $F(G)$ the subgroup generated by all its normal nilpotent subgroups. By Zorn's Lemma, any nilpotent subgroup of $G$ of nilpotency class $k$ is contained in a maximal such, which might not be unique.

**Lemma 10.** *There is a first order formula $\varphi_k$ in the language of groups such that, for any group $N$, $N$ is nilpotent of class $k$ if and only if $N \models \varphi_k$.*

*Proof.* Consider the formula

$$\forall x_1 \cdots \forall x_k [x_1, [x_2, [\cdots, [x_{k-1}, x_k] \cdots]]] = 1 \wedge \exists y_1 \cdots \exists y_{k-1} [y_1, [y_2, [\cdots, [y_{k-2}, y_{k-1}] \cdots]]] \neq 1.$$

□

**Theorem 11** (Łos)**.** *Let $(M_i)_{i \in \mathbf{N}}$ be a collection of $L$-structure, $\mathcal{U}$ an ultrafilter on $\mathbf{N}$ and $M$ the ultraproduct $\prod_i M_i \big/ \mathcal{U}$. One has $M \models \varphi$ if and only if $\{i \in \mathbf{N} : M_i \models \varphi\}$ is in $\mathcal{U}$.*

**Theorem 12** (Fitting subgroup of $G$)**.** *If the ultrafilter $\mathcal{U}$ is non-principal, for every natural number $k$, $G$ has a unique maximal normal nilpotent subgroup $N_k$ of nilpotency class $k$*

$$N_k = \prod_{n \in \mathbf{N}} \left( (\mathbf{Z}/p^n\mathbf{Z})^\times \cdot \mathbf{1} + p^{1 + \lfloor \frac{n-1}{k+1} \rfloor} \mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z}) \right) \Big/ \mathcal{U},$$

*hence the Fitting subgroup of $G$ is*

$$F(G) = \bigcup_{k=1}^{\infty} N_k.$$

*$F(G)$ is neither nilpotent, nor definable.*

*Proof.* Let $k$ be a fixed natural number. By Lemma 6 and Lemma 8, the normal subgroup

$$(\mathbf{Z}/p^n\mathbf{Z})^\times \cdot \mathbf{1} + p^{1 + \lfloor \frac{n-1}{k+1} \rfloor} \mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})$$

of $\mathrm{GL_m}(\mathbf{Z}/p^n\mathbf{Z})$ has nilpotency class $k$ for all but finitely many $n$. As $\mathcal{U}$ contains the Fréchet filter and as being of nilpotency class $k$ is expressible by a first order formula in the pure language of groups according to Lemma 10, by Łos Theorem, the ultraproduct

$$\prod_{n \in \mathbf{N}} \left( (\mathbf{Z}/p^n\mathbf{Z})^\times \cdot \mathbf{1} + p^{1 + \lfloor \frac{n-1}{k+1} \rfloor} \mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z}) \right) \Big/ \mathcal{U}$$

is a normal nilpotent subgroup of class $k$ of $G$. Reciprocally, if $g$ belongs to a normal nilpotent subgroup of class $k$, then $g^G$ generates a normal nilpotent subgroup of class at most $k$. By Łos Theorem, there is a set $I \in \mathcal{U}$ such that for all $n \in I$, the conjugacy class $g_n^{G_n}$ generates a nilpotent normal subgroup $\left\langle g_n^{G_n} \right\rangle$ of $G_n$ of class at most $k$. Let $n \in I$ be fixed. By Theorem 5, there is a unique natural number $1 \leqslant \ell \leqslant n$ such that

$$\mathrm{SL_m}(\mathbf{Z}/p^n\mathbf{Z}) \cap \left( \mathbf{1} + p^\ell \mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z}) \right) \subset \left\langle g_n^{G_n} \right\rangle \subset (\mathbf{Z}/p^n\mathbf{Z})^\times \cdot \mathbf{1} + p^\ell \mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z}).$$

As $\mathbf{1} + p^\ell \mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})$ is nilpotent of class $\left\lfloor \frac{n-1}{\ell} \right\rfloor$, one must have $k \geqslant \left\lfloor \frac{n-1}{\ell} \right\rfloor$. From Lemma 8, it follows that $\ell \geqslant \ell_{n-1}(k)$ for all but finitely many $n$ in $I$, so that $g$ belongs to the desired ultraproduct.

To show that the Fitting subgroup of $G$ is not definable, let $g_{n,\ell}$ be the elementary transvection $\mathbf{1} + p^\ell e_{ij}$ of $H_n$ for every $1 \leqslant \ell < n$. By Proposition 4, one has

$$\left\langle g_{n,\ell}^{G_n} \right\rangle = \mathrm{SL_m}(\mathbf{Z}/p^n\mathbf{Z}) \cap \left( \mathbf{1} + p^\ell \mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z}) \right),$$

hence

$$Z_q \left( \left\langle g_{n,\ell}^{G_n} \right\rangle \right) = \mathrm{SL_m}(\mathbf{Z}/p^n\mathbf{Z}) \cap Z_q \left( \mathbf{1} + p^\ell \mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z}) \right),$$

so $g_{n,\ell}^{G_n}$ generates a nilpotent subgroup of nilpotency class $\left\lfloor \dfrac{n}{\ell} \right\rfloor$. For every $\ell \geqslant 1$, let $n = k_n\ell + r_n$ where $k_n$ and $r_n$ denote the quotient and rest of the Euclidian division of $n$ by $\ell$ and let $g_\ell$ denote the class modulo $\mathcal{U}$ of

$$\left(1_{\mathrm{GL_m}(\mathbf{Z}/\mathbf{Z})}, 1_{\mathrm{GL_m}(\mathbf{Z}/p\mathbf{Z})}, \dots, 1_{\mathrm{GL_m}(\mathbf{Z}/p^\ell\mathbf{Z})}, g_{\ell+1,1}, g_{\ell+2,1}, \dots, g_{2\ell,2}, g_{2\ell+1,2}, \dots, g_{n,k_n}, \dots\right).$$

As $\left\lfloor \dfrac{n}{k_n} \right\rfloor = \ell$ holds for every $n \geqslant \ell^2$, the normal closure $\langle g_\ell\rangle^G$ is nilpotent of nilpotency class $\ell$ for every $\ell \geqslant 1$. Note that $G$ is $\aleph_1$-saturated by [Kei10, Theorem 5.6]. By [OH13, Theorem 1.3], the Fitting subgroup of $G$ is not definable. $\square$

**Lemma 13.** *There is a first order formula $\varphi_\ell$ in the language of groups such that, for any group $S$, $S$ is soluble of derived length $\ell$ if and only if $S \models \varphi_\ell$.*

*Proof.* Consider the term $t_\ell(x_1, \dots, x_{2^\ell})$ defined inductively by setting $t_1(x_1, x_2)$ to $[x_1, x_2]$ and $t_{\ell+1}(x_1, \dots, x_{2^{\ell+1}})$ to $[t_\ell(x_1, \dots, x_{2^\ell}), t_\ell(x_{2^\ell+1}, \dots, x_{2^{\ell+1}})]$. Then consider the formula

$$\forall x_1 \cdots \forall x_{2^\ell} t_\ell(x_1, \dots, x_{2^\ell}) = 1 \wedge \exists y_1 \cdots \exists y_{2^{\ell-1}} t_{\ell-1}(y_1, \dots, y_{2^{\ell-1}}) \neq 1.$$

$\square$

We call *soluble radical of* $G$ and write $R(G)$ the subgroup generated by all its normal soluble subgroups.

**Theorem 14** (soluble radical of $G$)**.** *If the ultrafilter $\mathcal{U}$ is non-principal, for every natural number $\ell$, $G$ has a unique maximal normal soluble subgroup $S_\ell$ of derived length $\ell$*

$$S_\ell = \prod_{n\in\mathbf{N}} \left((\mathbf{Z}/p^n\mathbf{Z})^\times \cdot \mathbf{1} + p^{\left\lceil \frac{n}{2^\ell} \right\rceil} \mathrm{M_m}(\mathbf{Z}/p^n\mathbf{Z})\right) \Big/ \mathcal{U},$$

*hence the soluble radical of $G$ is*

$$R(G) = \bigcup_{\ell=1}^{\infty} S_\ell = F(G).$$

*$R(G)$ is neither soluble, nor definable.*

*Proof.* By Lemma 7, Lemma 9, Lemma 13 and Łos Theorem, $S_\ell$ is a normal soluble subgroup of $G$ of derived length $\ell$. By Theorem 5 and Lemma 9, $S_\ell$ is maximal such. Note that $1 + \left\lfloor \frac{n}{2^\ell+1} \right\rfloor \leqslant \left\lceil \frac{n}{2^\ell} \right\rceil$ holds for every $n$ and $\ell$, so that one has $S_\ell \subset N_{2^\ell}$ hence $R(G)$ and $F(G)$ coincide. $\square$

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